TRINITY COLLEGE

YEAR 12
MATHEMATICS
SPECIALIST

Test 2, 2023
Section One: Calculator Free
Vectors in 3D

STUDENT'S NAME: Solutions [LAWRENCE]

DATE: Wednesday 10th May **TIME:** 16 minutes **MARKS**: 16
ASSESSMENT %: 10

INSTRUCTIONS:
Standard Items: Pens, pencils, drawing templates, eraser
Special Items:

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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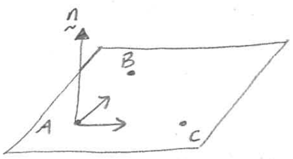
Question 1 **(10 marks)**

Consider the points $A = (1, 4, 7)$ $B = (-2, 4, 10)$ $C = (3, 4, 2)$ $D = (4, 0, 7)$

a) Determine the equation of the plane containing points A, B, C in the form $\boldsymbol{r}.\boldsymbol{n} = k$ (3 marks)



$\overrightarrow{AB} = B - A = \langle -3, 0, 3 \rangle$ ✓

$\overrightarrow{AC} = C - A = \langle 2, 0, -5 \rangle$

$\underset{\sim}{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \langle -3, 0, 3 \rangle \times \langle 2, 0, -5 \rangle = \langle 0, -9, 0 \rangle$ ✓

$\underset{\sim}{a} \cdot \underset{\sim}{n} = \langle 1, 4, 7 \rangle \cdot \langle 0, -9, 0 \rangle = -36$

$\therefore \underset{\sim}{r} \cdot \langle 0, -9, 0 \rangle = -36$ ✓

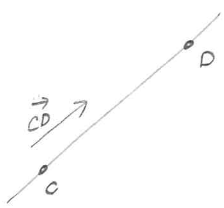
OR $\underset{\sim}{r} \cdot \langle 0, 1, 0 \rangle = 4$

✓ 2 vectors on plane

✓ cross product ($\underset{\sim}{n}$)

✓ equation of plane.

b) Determine the equation of the line going through points C and D. (2 marks)



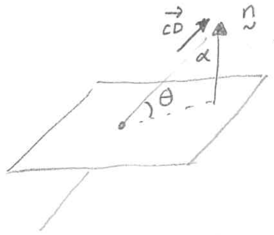
$\underset{\sim}{r} = \underset{\sim}{c} + \lambda \overrightarrow{CD} = \langle 3, 4, 2 \rangle + \lambda \langle 1, -4, 5 \rangle$

$\overrightarrow{CD} = D - C = \langle 1, -4, 5 \rangle$

✓ for $\overrightarrow{CD}$ or $\overrightarrow{DC}$

✓ equation of line

c) Determine an un-simplified expression for the angle between the plane and the line found in parts a) and b) respectively. (2 marks)



Angle between plane & line $= \theta$

$= 90 - \alpha$

$\vec{CD} \cdot \underset{\sim}{n} = |\vec{CD}||\underset{\sim}{n}| \cos \alpha$

$\langle 1, -4, 5 \rangle \cdot \langle 0, -9, 0 \rangle = (\sqrt{42})(9) \cos \alpha$

$36 = 9\sqrt{42} \cos \alpha$

$\alpha = \cos^{-1} \dfrac{36}{9\sqrt{42}}$

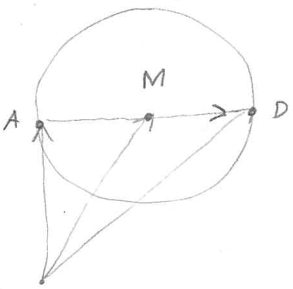
$\alpha = \cos^{-1} \dfrac{4}{\sqrt{42}}$

✓ use of dot product for α

$\theta = 90 - \cos^{-1} \dfrac{4}{\sqrt{42}}$

✓ $90 - \alpha$

d) Determine the equation of the sphere that has $\overrightarrow{AD}$ as its diameter. (3 marks)



$AD = D - A$

$= \langle 3, -4, 0 \rangle$

$|\vec{AD}| = \sqrt{9 + 16 + 0}$

$= 5$

$\therefore$ radius $= 2 \cdot 5$

OM = centre

$= \vec{OA} + \frac{1}{2} \vec{AD}$

$= \langle 1, 4, 7 \rangle + \frac{1}{2} \langle 3, -4, 0 \rangle$

$= \langle 1, 4, 7 \rangle + \langle \frac{3}{2}, -2, 0 \rangle$

$= \langle 5/2, 2, 7 \rangle$

$|\underset{\sim}{r} - \langle 5/2, 2, 7 \rangle| = 2 \cdot 5$

✓ for radius $\frac{1}{2}|\vec{AD}|$

✓ correct centre

✓ eqn of sphere

Question 2 **(6 marks)**

Consider the following system of equations

$$3x - 4y + z = 10$$
$$6x - 8y + kz = 20, \quad \text{where } k \in \mathbb{R}$$
$$2x + 8y - 2z = 4$$

a) If $k = 2$, describe the type of solution the system produces and state the geometric nature of the situation. (*Note, do not find the solution*). (2 marks)

$3x - 4y + z = 10$ $\downarrow \times 2$ = Same plane
$6x - 8y + 2z = 20$
$2x + 8y - 2z = 4$ → Different

✓ infinite along line

∴ Infinite solutions along a line
2 planes are parallel & same plane
third plane intersects these 2 along a line.

✓ description of $\underline{\underline{3}}$ planes.

b) If $k = 1$,

i) solve the system of equations. (3 marks)

$$\left[\begin{array}{ccc|c} 3 & -4 & 1 & 10 \\ 6 & -8 & 1 & 20 \\ 2 & 8 & -2 & 4 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 6 & -8 & 2 & 20 \\ 6 & -8 & 1 & 20 \\ 6 & 24 & -6 & 12 \end{array}\right] \begin{array}{l} R_1 \times 2 \\ \\ R_3 \times 3 \end{array}$$

$$\left[\begin{array}{ccc|c} 6 & -8 & 2 & 20 \\ 0 & 0 & -1 & 0 \\ 0 & 32 & -8 & -8 \end{array}\right] \begin{array}{l} \\ R_2 - R_1 \\ R_3 - R_1 \end{array}$$

∴ $-z = 0$
$z = 0$

$32y - 8(0) = -8$
$32y = -8$
$y = -1/4$

$6x - 8(-1/4) + 2(0) = 20$
$6x + 2 = 20$
$6x = 18$
$x = 3$

✓ for 1 row red.
✓ for 2 row red.
✓ for (x, y, z)
$(3, -1/4, 0)$

ii) State the geometric interpretation of your result to b i). (1 mark)

3 planes intersect at a unique point.

✓ unique pt.

END OF QUESTIONS

TRINITY COLLEGE

YEAR 12
MATHEMATICS
SPECIALIST

Test 2, 2023
Section Two: Calculator Allowed
Vectors in 3D

STUDENT'S NAME: ______________________________

DATE: Wednesday 10th May **TIME:** 35 minutes **MARKS:** 32
ASSESSMENT %: 10

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser
Special Items: 1 A4 page notes, Classpad, Scientific Calculator

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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Question 3 **(9 marks)**

For this question, assume gravity does not exist – This is not a projectile motion question. Also assume that the tip of the arrow is being defined by the vector function.

An archer, A, fired an arrow with the following vector function representing the arrow's position, where t represents seconds after being fired.

$$\overrightarrow{OA_t} = \begin{bmatrix} 5+t \\ 5-2t \\ 5-0.5t^2 \end{bmatrix}$$

A spherical balloon has the vector equation $\left| \boldsymbol{r} - \begin{bmatrix} 7 \\ 1 \\ 3 \end{bmatrix} \right| = 1$.

a) Determine the cartesian equation of this spherical balloon. (1 mark)

$$(x-7)^2 + (y-1)^2 + (z-3)^2 = 1$$

✓ cartesian eqn

(W) correct radius

b) The arrow first hit the balloon 1.654 seconds after being fired. Show how this time value was calculated, and state the time solution to 4 decimal places. (3 marks)

$$\left| \begin{pmatrix} 5+t \\ 5-2t \\ 5-0.5t^2 \end{pmatrix} - \begin{pmatrix} 7 \\ 1 \\ 3 \end{pmatrix} \right| = 1$$

✓ sub line into sphere

$$\sqrt{(-2+t)^2 + (4-2t)^2 + (2-0.5t^2)} = 1$$

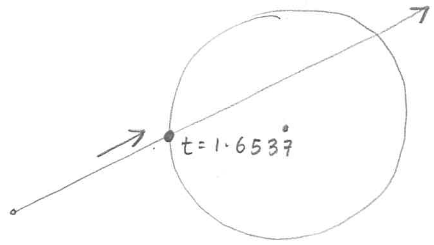
✓ for showing $\sqrt{\quad} = 1$

Use C.A.S to solve

$t = 1.6537$ seconds

✓ 1.6537

c) Calculate the co-ordinate where the arrow first hit the balloon. (1 mark)



$$\overrightarrow{OA_t} = \begin{bmatrix} 5 + 1.6537 \\ 5 - 2(1.6537) \\ 5 - 0.5(1.6537)^2 \end{bmatrix}$$

$$= (6.65,\ 1.69,\ 3.63)$$

✓ sub in 1·654 or 1·6537

d) Determine the speed of the arrow when it first hit the balloon. (2 marks)

$$V_A(t) = \frac{d\,\overrightarrow{OA_t}}{dt} = \begin{bmatrix} 1 \\ -2 \\ -t \end{bmatrix}$$

✓ velocity

$$V_A(1.6537) = \begin{bmatrix} 1 \\ -2 \\ -1.6537 \end{bmatrix}$$

✓ $|v|$

$$|V_A(1.6537)| = \sqrt{1 + 4 + (1.6537)^2} = 2.78 \text{ m/s}$$

e) Determine the total distance travelled from when it was fired to when it first hit the balloon.

The distance formula is $\int_a^b |v(t)|dt$ (2 marks)

$$\int_0^{1.6537} |V_A(t)|\,dt = \int_0^{1.6537} \sqrt{1 + 4 + (-t)^2}\ dt$$

✓ correct integral ⓦ boundaries

$$= 4.01 \text{ m}$$

✓ — m

Question 4 **(10 marks)**

On the moon, gravity accelerates all things downward at $1.62 m/s^2$.

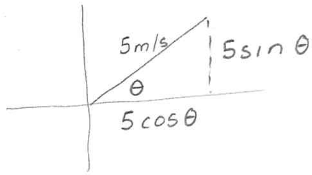
A machine can launch rocks at $5 m/s$ at an angle of $\theta°$ to the positive x axis from the origin.

a) Show how the displacement of the rock after t seconds can be shown by

$$\boldsymbol{r} = \begin{bmatrix} 5\cos(\theta)\, t \\ -0.81t^2 + 5\sin(\theta)\, t \end{bmatrix} \text{ meters}$$

(3 marks)

$a(t) = \begin{pmatrix} 0 \\ -1.62 \end{pmatrix}$



$\underset{\sim}{v}(t) = \int \underset{\sim}{a}(t)\, dt = \int \begin{pmatrix} 0 \\ -1.62 \end{pmatrix} dt$

$= \begin{bmatrix} 0 \\ -1.62t \end{bmatrix} + C$

$\underset{\sim}{V_0} = \begin{bmatrix} 5\cos\theta \\ 5\sin\theta \end{bmatrix}$

$= \begin{bmatrix} 5\cos\theta \\ 5\sin\theta - 1.62t \end{bmatrix}$

$\underset{\sim}{r}(t) = \int \underset{\sim}{v}(t)\, dt = \int \begin{bmatrix} 5\cos\theta \\ 5\sin\theta - 1.62t \end{bmatrix} dt$

$\underset{\sim}{r_0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$= \begin{bmatrix} (5\cos\theta)t \\ (5\sin\theta)t - 0.81t^2 \end{bmatrix} + C$

$= \begin{bmatrix} (5\cos\theta)t \\ (5\sin\theta)t - 0.81t^2 \end{bmatrix}$

✓ a(t)

✓ v(t) with c

✓ r(t) with c

On a flat ground, a rock was launched at 56° to the positive x axis.

b) Calculate when the rock was at its highest point. (2 marks)

$\underset{MAX}{}$

$\theta = 56°$

$\underset{\sim}{r_t} = \begin{bmatrix} (5\cos 56°)t \\ (5\sin 56°)t - 0.81t^2 \end{bmatrix} \begin{matrix} \rightarrow \\ \uparrow \end{matrix}$

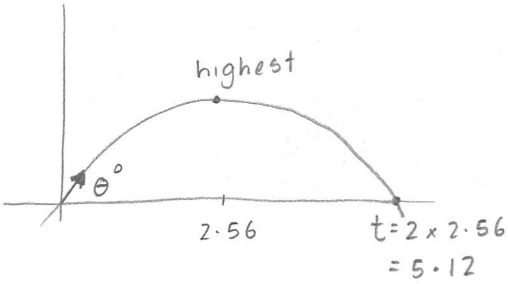
$f_{max}\left((5\sin 56°)t - 0.81t^2,\ t\right)$

✓ uses j component

✓ 2.56 secs

$t = 2.56$ seconds

c) Calculate the horizontal distance the rock travelled when it first hit the ground. (3 marks)



✓ correct t

✓ subs t into x component

✓ __ m

$$\underset{\sim}{r}_t(5.12) = \begin{bmatrix} (5\cos 56°)(5.12) \\ 0 \end{bmatrix} = \begin{bmatrix} 14.32 \\ 0 \end{bmatrix}$$

$$= 14.32\text{ m}$$

d) Using the fact that fMax(sin(α) cos(α)) occurs when $\alpha = 45°$, prove that launching at an angle of 45° to the positive x axis will lead to the rock travelling the furthest horizontal distance. (2 marks)

Horizontal component $= -0.81t^2 + (5\sin\theta)t$

Hits ground when $-0.81t^2 + (5\sin\theta)t = 0$

$t(5\sin\theta - 0.81t) = 0$

$t = 0$ OR $5\sin\theta - 0.81t = 0$

$t = \dfrac{5\sin\theta}{0.81}$

Horizontal distance $\Rightarrow (5\cos\theta)t$

✓ solving for t

$\therefore 5\cos\theta\left(\dfrac{5\sin\theta}{0.81}\right)$

✓ t into → distance

$= \dfrac{25}{0.81}(\cos\theta\sin\theta) \rightarrow$ Maximised when $\theta = 45°$

$\therefore$ Horizontal distance maximised when $\theta = 45°$

Question 5 **(8 marks)**

A hydraulic press can create a huge crushing force between two flat surfaces.
Typically, the bottom plate is fixed whilst the top plate moves.

Consider two parallel planes which represent the two parallel surfaces of the hydraulic press:

$$r.\begin{bmatrix}3\\5\\6\end{bmatrix}=2 \quad \text{and} \quad r.\begin{bmatrix}6\\10\\12\end{bmatrix}=k, \quad k\geq 4$$

a) Explain mathematically why these two surfaces are parallel, and explain which plane represents the top, moving plate. (1 mark)

$P_1 \quad \underset{\sim}{n} = \begin{bmatrix}3\\5\\6\end{bmatrix}$

$P_2 \quad \underset{\sim}{n} = \begin{bmatrix}6\\10\\12\end{bmatrix} = 2\begin{bmatrix}3\\5\\6\end{bmatrix}$ ✓

$\therefore$ normals are parallel
$\therefore$ planes are parallel

$P_2 \Rightarrow \underset{\sim}{r} \cdot \begin{bmatrix}6\\10\\12\end{bmatrix} = k$

this plane is moving as k is a changing variable ✓

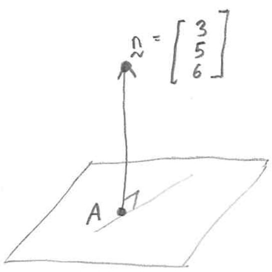
Explanation of both needed for 1 mark.

A spherical marble of diameter 10cm is trapped between the plates but has not yet been crushed.

b) Determine any possible vector equation for a sphere representing a marble of diameter 10cm trapped between the two planes. (4 marks)

A Point on plane $= \begin{pmatrix}-1\\1\\0\end{pmatrix}$

* could make any number of points.



$\therefore$ We need a point 5 units away from A, travelling along $\begin{bmatrix}3\\5\\6\end{bmatrix}$ ($\perp$ to plane)

Centre of sphere $= \begin{bmatrix}-1\\1\\0\end{bmatrix} + 5\left(\frac{1}{\sqrt{70}}\right)\begin{bmatrix}3\\5\\6\end{bmatrix}$

$= \begin{bmatrix}0.79\\3.99\\3.59\end{bmatrix}$

$\left|\left|\begin{bmatrix}3\\5\\6\end{bmatrix}\right|\right| = \sqrt{9+25+36}$

$= \sqrt{70}$

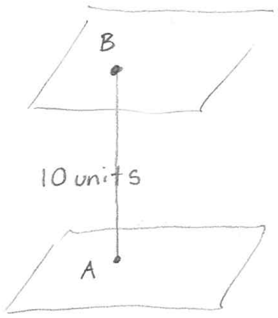
$= 8.367$

$\left|\left|\widehat{\begin{bmatrix}3\\5\\6\end{bmatrix}}\right|\right| = \frac{1}{\sqrt{70}}$

$\therefore \left|\underset{\sim}{r} - \begin{bmatrix}0.79\\3.99\\3.59\end{bmatrix}\right| = 5$

✓ pt on plane
✓ unit vector of $\underset{\sim}{n}$
✓ centre
✓ eqn of sphere

c) Determine the value of k. (3 marks)



$\therefore$ We need a point 10 units away from A, travelling along $\begin{bmatrix}3\\5\\6\end{bmatrix}$ ($\perp$ to plane)

$$B = \begin{bmatrix}-1\\1\\0\end{bmatrix} + 10\left(\frac{1}{\sqrt{70}}\right)\begin{bmatrix}3\\5\\6\end{bmatrix}$$

$$B = \begin{pmatrix}2.59\\6.98\\7.17\end{pmatrix}$$

✓ pt B

$$\begin{pmatrix}2.59\\6.98\\7.17\end{pmatrix} \cdot \begin{pmatrix}6\\10\\12\end{pmatrix} = k$$

✓ sub B into eqn of plane.

✓ k value.

$\therefore k = 171.33$

Question 6 **(5 marks)**

A fly is travelling with a position vector of $OF_t = \begin{bmatrix} 3+2t \\ -3+t \\ 7+0.5t \end{bmatrix}$ and a wasp is hunting it with a position vector of $OW_t = \begin{bmatrix} 8+1.2t \\ 15-1.88t \\ 4+ae^t \end{bmatrix}$.

a) If there is a collision, determine a and the time when the collision occurs. (2 marks)

$OF_t = OW_t$ $\qquad 3+2t = 8+1.2t$

$t = 6.25$

✓ t

$4 + ae^{6.25} = 7 + 0.5(6.25)$ ✓ a

$a = 0.0118$

b) Determine the initial velocity of the wasp. (1 mark)

$V_{OW_t} = \begin{pmatrix} 1.2 \\ -1.88 \\ 0.0118e^t \end{pmatrix} \quad \therefore \quad V_{W(0)} = \begin{pmatrix} 1.2 \\ -1.88 \\ 0.0118 \end{pmatrix}$

✓ V_0

When hunting, a dragonfly predicts the future location of it's target and flies in a straight line to collide, travelling at a constant $3m/s$.

(c) A dragonfly is also hunting the fly. If the dragonfly starts at the location (1, 5, 3), determine a system of 4 linear equations that would be solved to find the velocity of the dragonfly and the time of collision between the dragonfly and the fly.
DO NOT SOLVE THE EQUATIONS. (2 marks)

$OD_t = \begin{bmatrix} 1+at \\ 5+bt \\ 3+ct \end{bmatrix} \quad \therefore \quad a^2+b^2+c^2 = 3^2$ ✓ magnitude equation

$3+2t = 1+at$ ✓ 3 eqns from i, j, k components

$-3+t = 5+bt$

$7+0.5t = 3+ct$

END OF QUESTIONS